JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **200,** 382–387 (1996)
ARTICLE NO. 0212

A Remark on the Nevanlinna-Pólya Theorem in Analytic Function Theory

Hiroshi Haruki

Department of Pure Mathematics, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

and

Themistocles M. Rassias

4, Zagoras Street Paradissos, Amaroussion, 15125 Athens, Greece

Submitted by H. M. Srivastava

Received July 10, 1995

We provide an extension of the Nevanlinna–Pólya theorem for two analytic functions of a complex variable.

The Nevanlinna–Pólya theorem reads (cf. [1, 2]):

THEOREM 1. *Let n be an arbitrarily fixed positive integer and, for each k $(k = 1, 2, 3, \ldots, n)$, let f_k and g_k be analytic functions of a complex variable z on a nonempty domain D. If f_k and g_k $(k = 1, 2, 3, \ldots, n)$ satisfy*

$$\sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2 \tag{1}$$

on D and if $f_1, f_2, f_3, \ldots, f_n$ are linearly independent on D, then there exists an $n \times n$ unitary matrix C, where each of the entries of C is a complex

0022-247X/96 $18.00

constant such that

$$\begin{pmatrix} g_1(z) \\ g_2(z) \\ g_3(z) \\ \vdots \\ g_n(z) \end{pmatrix} = C \cdot \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ \vdots \\ f_n(z) \end{pmatrix} \tag{2}$$

holds on D.

We consider now the case when $n = 2$ in the above theorem.

We shall prove that the above theorem still holds even if we omit the hypothesis that f_1, f_2 are linearly independent on D.

THEOREM 2. *Let, for each* k $(k = 1, 2)$, f_k *and* g_k *be analytic functions of a complex variable* z *on a nonempty domain* D. *If* f_k *and* g_k $(k = 1, 2)$ *satisfy*

$$|f_1(z)|^2 + |f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2 \tag{3}$$

on D, *then there exists a* 2×2 *unitary matrix* C *where each of the entries of* C *is a complex constant such that*

$$\begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix} = C \cdot \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \tag{4}$$

holds on D.

Proof. We consider the following two cases:

Case A. If f_1, f_2 are linearly independent on D, the proof is clear from the Nevanlinna–Pólya theorem.

Case B. If f_1, f_2 are linearly dependent on D, there exist two complex constants c_1, c_2, at least one of which is nonzero such that

$$c_1 f_1(z) + c_2 f_2(z) = 0 \tag{5}$$

holds on D.

We discuss two subcases:

Case B_1. Let $c_2 \neq 0$. In this case, by (5) we obtain that

$$f_2(z) = -\frac{c_1}{c_2} f_1(z) \tag{6}$$

holds on D. If we set $b = -c_1/c_2$, then, by (6) we have on D

$$f_2(z) = bf_1(z). \tag{7}$$

Substituting (7) back into (3) yields on D

$$(1 + |b|^2)|f_1(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2. \tag{8}$$

We may assume that $f_1 \not\equiv 0$ on D. Otherwise the proof is trivial. Since, by hypothesis, f_1 $(\not\equiv 0)$ is analytic on D, f_1 $(\not\equiv 0)$ is continuous on D. Hence, there exists a nonempty subdomain D_1 on D, where $f_1(z) \neq 0$. So, by (8) we obtain on D_1

$$\left|\frac{g_1(z)}{f_1(z)}\right|^2 + \left|\frac{g_2(z)}{f_1(z)}\right|^2 = 1 + |b|^2. \tag{9}$$

Taking the Laplacians $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ of both sides of (9) with respect to $z = x + iy$ $(x, y$ real) yields on D_1

$$\left|\left(\frac{g_1(z)}{f_1(z)}\right)'\right|^2 + \left|\left(\frac{g_2(z)}{f_1(z)}\right)'\right|^2 = 0, \tag{10}$$

since $\Delta|p(z)|^2 = 4|p'(z)|^2$ (cf. [3, p. 94]), where p is an analytic function of z. By (10) we have on D_1

$$\left(\frac{g_1(z)}{f_1(z)}\right)' = 0, \qquad \left(\frac{g_2(z)}{f_1(z)}\right)' = 0$$

and, therefore,

$$g_1(z) = c \cdot f_1(z), \qquad g_2(z) = d \cdot f_1(z), \tag{11}$$

where c, d are complex constants.

By the identity theorem, (11) holds on D. Substituting (11) back into (9) yields

$$|c|^2 + |d|^2 = 1 + |b|^2. \tag{12}$$

If we set

$$U \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{1}{\sqrt{1+|b|^2}} & -\dfrac{\bar{b}}{\sqrt{1+|b|^2}} \\ \dfrac{b}{\sqrt{1+|b|^2}} & \dfrac{1}{\sqrt{1+|b|^2}} \end{pmatrix} \tag{13}$$

and

$$V \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{c}{\sqrt{1+|b|^2}} & -\dfrac{\bar{d}}{\sqrt{1+|b|^2}} \\ \dfrac{d}{\sqrt{1+|b|^2}} & \dfrac{\bar{c}}{\sqrt{1+|b|^2}} \end{pmatrix}, \tag{14}$$

then it is easy to prove, by using the definitions of a unitary matrix and multiplication of two 2×2 matrices, that

$$U \cdot \begin{pmatrix} \sqrt{1+|b|^2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix} \tag{15}$$

and

$$V \cdot \begin{pmatrix} \sqrt{1+|b|^2} \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{16}$$

hold.

We set

$$C = V \cdot U^{-1}. \tag{17}$$

Since all 2×2 unitary matrices form a group under the standard multiplication of matrices, by (17), C is a 2×2 unitary matrix.

Next we shall prove that (4) holds on D. By (15) we obtain

$$U^{-1} \cdot \begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{1+|b|^2} \\ 0 \end{pmatrix}. \tag{18}$$

Thus we have on D that

$$\begin{aligned} C \cdot \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} &= f_1(z) \cdot C \cdot \begin{pmatrix} 1 \\ b \end{pmatrix} && (\text{by } (7)) \\ &= f_1(z) \cdot V \cdot \left(U^{-1} \cdot \begin{pmatrix} 1 \\ b \end{pmatrix} \right) && (\text{by } (17)) \\ &= f_1(z) \cdot V \cdot \begin{pmatrix} \sqrt{1 + |b|^2} \\ 0 \end{pmatrix} && (\text{by } (18)) \\ &= f_1(z) \cdot \begin{pmatrix} c \\ d \end{pmatrix} && (\text{by } (16)) \\ &= \begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix} && (\text{by } (11)). \end{aligned}$$

Hence (4) holds on D. Thus in this case the proof of the theorem is now completed.

Case B_2. Let $c_2 = 0$ and $c_1 \neq 0$. In this case, by (5) we obtain $f_1 \equiv 0$. Hence, by (3)

$$|f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2 \tag{19}$$

holds on D.

By (19) and by a similar discussion to that of Case B_1 (b becomes 0) we obtain the desired result. Q.E.D.

Remark. D'Angelo [1, p. 102] proved the following generalized result.

PROPOSITION 3. *Suppose that B is an open ball about O in $\mathbf{C}^q$ and that F,G are holomorphic mappings from B to $\mathbf{C}^N$ for which*

$$\|F\|^2 = \|G\|^2.$$

Then there exists $U \in \mathbf{U}(N)$ for which $F = UG$.

($\mathbf{U}(N)$ denotes the group of unitary matrices on $\mathbf{C}^N$.)

The method of proof of the present paper is quite different from that given by D'Angelo [1]. The heart of the method of our proof is Eq. (10).

REFERENCES

1. J. P. D'Angelo, ''Several Complex Variables and the Geometry of Real Hypersurfaces,'' CRC Press, Boca Raton, 1993.
2. R. Nevanlinna and G. Pólya, Unitäre Transformationen Analytischer Funktionen, *Jahresbericht der Deutschen Mathematiker-Vereiningung* **40** (1931), 80 (Aufgabe 103).
3. G. Pólya and G. Szegö, ''Aufgaben und Lehrsätze aus der Analysis I,'' Springer-Verlag, Berlin, 1954.